Liquidity, firm value, and long-term bank relationships*

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Abstract

Our interest here concerns liquidity supply as a distinctive feature of the bank-firm. relationship. Any agent facing an opportunity or a commitment may find him/herself unexpectedly illiquid, and hence he/she may find it profitable to borrow "on call" if this costs less than missing the opportunity or defaulting on the commitment, or costs less than using non-money goods as means of payment. We first examine the relation between firm value and access to liquidity supply, and then we investigate the efficiency properties of bank refinancing contracts in a continuous time stochastic model of a repeated bank-firm relationship where the key problem is the credibility of the mutual commitment between the two parties. Our main finding is that efficient, i.e. cost-minimizing, and renegotiation-proof contracts emerge in the absence of perfect commitment and enforceability of payments as the borrower and the bank can extert *mutual threat of termination.*

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1 Introduction

The first principle of the theory of financial intermediation is that any private borrower-lender relationship may face severe efficiency limits in dealing with transaction costs, risk management, agency problems and liquidity requirements. Solutions to these efficiency problems elicit the rise of intermediaries. In this paper we focus on liquidity as a key financial problem and a distinctive feature of firm-bank intermediation.

Liquidity is a difficult economic concept: it is only relevant in relation to transaction costs and uncertainty. As a first approximation, it means availability of means of payment. at (almost) zero transaction-costs. In principle all goods might be used as means of payment against other goods, but the existence of *different* transaction costs may make some goods more liquid than others. Transactions in modern market economies rely upon the existence of an artificial good called "money" created to be the *perfectly liquid means of payment.* Lack of money in the presence of a profitable transaction or of a payment commitment entails a loss because the transaction cannot be done or the commitment cannot be honoured, or because of the costs of using non-money goods as means of payment. Thus no rational agent would plan to be illiquid: yet illiquidity comes as an expected lack of money when payments and/or receipts flows are uncertain. The costs of illiquidity are a well-known explanation of the reason why rational agents are willing to hold money as an apparently worthless asset, on the one hand, and of the existence of specialized agents who are able to manage liquidity efficiently on the other. Efficiency in liquidity management may increase efficiency for the economy as a whole in two main ways: 1) by offering deposit services to a large number of money holders with uncorrelated liquidity needs so that total "idle" deposits can be less than the sum of individual money holdings, 2) by offering liquidity "on call" so that agents are allowed to economize their individual holdings of the worthless asset. These two services have long been viewed as a key explanation, in terms of social efficiency gain, of the emergence of bank intermediation (see e.g. Hicks, 1967, 1989; Goodhart, 1989; and the review by Freixas and Rochet, 1997, ch.2). Thus, even though in practice liquidity problems may sometimes be resolved without banks (firms' lack of liquidity is often overcome by owners or by means of equity capital increases or non-bank loans), banks are typically the specialized institutions in liquidity supply in modern economies.

In spite of its key importance, the role of liquidity in theories of firm valuation and management, investment valuation and firm-bank relationship is not yet treated systematically. Two paradigmatic examples that we shall discuss are, on the one hand, standard firm and investment valuation analysis based on intertemporal optimization, which implicitly assumes that in all states of unexpected excess payments any agent can freely lend or borrow at the market interest rate or, as a special case, can obtain liquidity at no cost. On the other hand, one can find the opposite extreme case represented by the so-called "standard debt contract" in bank theory (see e.g. Freixas and Rochet, 1998, ch.4), which implies that a borrower can receive no additional liquidty as soon as he/she is unable to meet his/her debt obligation. It is often recognized that both extreme cases should, to say the least, only be taken as a first approximation, but the liquidity-supply part of the problem is generally not developed.

Elements for analysis of the role of liquidity for firms and in bank-firm relationships are dispersed in bank and corporate finance theories. In the development of firm value analysis, the event of bankruptcy for levered firms has been introduced and quantified (e.g. Merton, 1974; Black and Cox, 1976; Brennan and Schwartz, 1978; Leland, 1994). Bankruptcy can be seen as the extreme consequence of illiquidity. Since these models show that bankruptcy affects firm value, they also imply that liquidity supply, as an alternative to bankruptcy, does. However, standard models of firm valuation typically take bankruptcy as a state defined *a priori* by exogenous parameters (e.g. Brennan and Schwartz, 1978), hence they do not consider that exerting bankruptcy may in most of the cases be a *choice* either of the debtor or of the creditor. On the one hand, the debtor may be technically solvent but may choose not to pay an obligation; on the other hand, a technically insolvent debtor can be refinanced by an injection of liquidity instead of being declared bankrupt. Leland (1994) endogenizes the firm value that triggers bankruptcy as an optimal firm's choice, but he does not examine creditors' behaviour.

The bankruptcy vs. refinancing choice, possibly in the context of debtor-creditor strategic interaction, has been matter of analysis in the studies on strategic debt repayment and incentives to repay (e.g. Eaton and Gersovitz, 1981; Allen, 1983; Stiglitz and Weiss, 1983; Bolton and Scharfstein, 1990), and on corporate debt restructuring and refinancing plans (e.g. Anderson and Sundaresan, 1996, Mella-Barral and Perraudin, 1997, Perraudin and Psillaki, 1999). The two strands, however, have specialized themselves on two separate sides of the problem: bankruptcy as an incentive-to-repay device, on the one hand, under the implicit assumption that bankruptcy is unconditionally exerted when the debtor does not pay; refinancing as a creditor's once-for-all option, on the other, under the implict assumption that the debtor will pay his/her new obligation thereafter. Yet the two sides of the problem are significantly intertwined: when the creditor (or possibly a third liquidity supplier) considers the opportunity to refinance the debtor, he/she should also assess the debtor's incentive to repay thereafter; at the same time, if a choice opportunity exists between bankruptcy and refinancing, the role of bankruptcy as incentive to repay may be jeopardized. Moreover, the two strands of literature have so far tended to use different theoretical setups and analytical tools that make comparison and integration difficult: studies on strategic debt repayment and incentives to repay have mainly used debtor-creditor models in discrete finite time (typically in two or three periods time horizon) with no explicit firm value analysis, whereas studies on corporate debt restructuring and refinancing plans have mainly been cast in stochastic continuous-time firm value analysis.

In this paper we provide a unified framework for analysis of the role of liquidity in firm value, and of different conditions and efficiency effects whereby liquidity can be supplied as an alternative to bankruptcy. To address this problem we use a highly stylized model of a firm with an infinetely lived investment project characterized by a (minimum) payment commitment towards investors at each point in time, a stochastic cash flow (profit) following a trendless geometric Brownian motion, and a non-negative net present value of the project (note that the committed agent might as well be a household with cash flow given by income, or a sovereign state with cash flow given by its trade balance, etc.). Hence the firm faces a liquidity problem as soon as the cash flow falls short of the payment commitment. All agents are risk neutral and fully informed.

We wish to signal three simplifying features of our model. First, liquidity supply, like any financial transaction, should be based on an assessment of the future prospects of the recipient: illiquidity may be the result of a transitory unexpected event in an otherwise profitable investment, or may be a signal of unprofitability. We shall work with a stochastic process such that the liquidity problem is perceived as *a temporary shortage of cash*: in fact, though the observed level of cash flow is the best expectation of the future levels, a positive probability exists that the cash flow recovers (the initial net present value of the investment is indeed non-negative).[1] Second, we shall rely on

[1]As the chash flow is a Markov process in levels, possible liquidity injections can in fact

the above mentioned efficiency arguments and we shall simply assume that the liquidity service is exclusive of banks.Third, to focus on the problem at hand we shall also assume that the firm has non-money assets that can be liquidated at a cost but no worthless money holdings, which can however be replaced by liquidity supply "on call" from a bank.

In the first place, in section 2 we shall address the problem of the role of liquidity in firm valuation. The point is that firm value is not independent of conditions of liquidity supply throughout the firm's life. We shall see that the standard measure of financial value (the excess of the expected cash-flow present value over the investment cost) is a correct measure only if *infinite liquidity supply* exists, i.e. only if each time the cash flow is less than the payment commitment, the firm can receive liquidity at no cost. By contrast, we consider the opposite case of *zero liquidity supply*, i.e. the first time the cash flow falls short of the payment commitment the firm is liquidated. Truncation of the firm's life at the first illiquidity state entails a loss of value measured by the expected liquidation cost of non money-assets (recall that the initial net present value of the project is non negative). This amount also provides an indirect measure of the value of liquidity to the firm. Note that, as we shall argue, these general principles hold independently of the nature of the payment commitment (dividends, interests, etc.) and of the kind of the recipient subject (shareholders, banks, etc.).

In section 3 we shall remove both the infinite and zero liquidity-supply cases and shall investigate the case of costly liquidity supply. Following previous considerations, we shall treat liquidity supply within a "bank refinancing contract" (BRC) specifying the terms and cost of refinancing conditional on illiquidity, and we shall frame it into long-term contracts analysis and more generally within the literature that relates the specific advantages of the bank-borrower relationship in the long-term, personal, non-marketable nature of this relationship (von Thadden, 1990; Hellwig, 1991; Mayer, 1994). A significant part of this literature aims at showing how long-term bank-borrower relationships resolve asymmetric information problems and agency problems. Within that framework, it has been argued that long-term ties can also induce inefficiency since the bank can have the opportunity to charge extra-costs to "informationally captured" firms (Sharpe, 1990). Here we do not introduce informational imperfections; liquidity problems do not have to do with informational imperfections but with plain uncertainty and transac-

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be based on corrent observations only.

tion costs. We wish to examine how the repeated relationship between bank and firm may solve the problem of designing an efficient BRC in a setup where: 1) since liquidity affects the firm's value, the cost of liquidity has distributional effects between firm and bank, 2) once the BRC is signed, the bank is committed to refinance the firm in the future whenever the contract conditions apply, and the firm is commited to repay the bank thereafter, 3) the mutual commitment between the two parties must be "renegotiation-proof" (for this aspect of contract theory see Salanié, 1997).

As a first step we assume perfect commitment, and we find a whole set of feasible values of the marginal cost of liquidity that can be charged by the bank up to an upper bound ("liquidity premium"). Liquidity injections allow the firm to carry on his/her initial project, but the "liquidity premium" redistributes part of the project's value from the firm to the bank; hence this result can be viewed as an extension of the idea that long-term relationships can solve specific borrower-lender problems only at a cost. Then we remove perfect commitment and address three core issues in long-term contract analysis. 1) *Non private enforceability* or the borrower's willingness-to-pay problem (Bolton-Scharfstein, 1990; Haubrich, 1989; Hart-Moore, 1996, sec.4): after the BRC has been signed, and even though the borrower's state is freely observable, the bank may be unable to force the borrower to pay unless payment is enforced by an external legal authority; sure payments are only those based on the borrower's willingness to pay. 2) *Renegotiation*: a long-term contract may be "non renegotiation-proof" (Salanie', 1997), i.e. the borrower, once he/she finds him/herself in specific states, may, in alternative to paying the contractual sum and in order to avoid bankruptcy, induce the bank to accept a renegotiation of the contractual terms. 3) *Threat of termination*: as a consequence of the previous two points, it is generally argued that the bank's "threat of termination" is an essential part of the contractual equilibrium (Allen, 1983; Stiglitz-Weiss, 1983; Bolton-Scharfstein, 1990). The analytical framework that we have chosen allows us to deal with all three issues in a coherent and general way consistent with current firm value analysis.Our main finding is that efficient (cost-minimizing) and "renegotiation-proof" BRCs do emerge in the absence of perfect commitment as the borrower and the bank can exert *mutual threat of termination*.

2 Firm value and liquidity supply

We start by examining the role of liquidity supply in firm valuation. Take as a benchmark a standard intertemporal optimal programme: at each point in time the agent should be able to realize the relevant time step of the programme, which implies that he/she should dispose of the money value of the programme at each point in time. Now introduce uncertainty: at time 0 the agent *expects* he/she will be able to realize his/her planned optimal step at each future time, which implies that the agent *expects* he/she will dispose of the relevant money value. What if this latter expectation is violated *in any particular point in time* (suppose that the agent's cash flow is a stochastic variable following any probability law: by Tchebycheev inequality the probability that any single realization differs from the expected value by any arbitrary number is strictly positive)? This part of the problem is not made explicit in standard intertemporal optimization; an auxiliary assumption is present: the agent can freely lend or borrow at a single market interest rate *vis-à-vis* any unexpected positive or negative excess of liquidity (under the usual end-of-time condition of zero present value of lending and borrowing); as a particular case, the market interest rate for liquidity can be zero, i.e. there exists *infinite liquidity supply.*

2.1 Analytical framework

Let us consider a risk neutral agent such as a firm which owns an infinitely-lived investment project of initial amount I, illiquid non-money assets $K \leq I$, $K = kI$, $k \in (0, 1]$, and no worthless money balances. The investment project generates instantaneous profits π_t which are uncertain and driven by a trendless geometric Brownian motion.

$$d\pi_t = \sigma \pi_t dW_t, \qquad \text{with } \pi_0 = \pi > 0,\ \sigma > 0 \tag{1}$$

where dW_t is the standard increment of a Wiener process (or Brownian motion), uncorrelated over time and satisfying the conditions that $E(dW_t) = 0$ and $E(dW_t^2) = dt$. Therefore $E(d\pi_t) = 0$ and $E(d\pi_t^2) = (\sigma\pi_t)^2 dt$, i.e. starting from the initial value π_0, the random position of the istantaneous profits π_t at time $t > 0$ has normal distribution with mean π_0 and variance $\pi_0^2(e^{\sigma^2 t} - 1)$ which increases as we look further and further in the future.

Assumption 1. A constant market interest rate r is given. By Modigliani-Miller theorem, the minimum no-arbitrage constant payment b due to investors is

$$\int_0^\infty e^{-rt} b dt \equiv \frac{b}{r} = I \qquad (2)$$

The project can be undertaken.

By the standard net present value (NPV) technique the project's value for the firm, $V^l(\pi; b)$ can be expressed by (e.g. Harrison, 1985, pag.44):

$$V^l(\pi; b) = E_0 \left\{ \int_0^\infty e^{-rt} \pi_t dt \mid \pi_0 = \pi \right\} - I = \frac{\pi}{r} - I \geq 0 \ , \qquad (3)$$

where $E_0(.)$ denotes expectation given available information at time 0.

Using the no-arbitrage condition (2), equation (3) can be re-written as

$$V^l(\pi; b) = \frac{\pi - b}{r} \geq 0 \ , \qquad (4)$$

whose meaning is that the project's NPV is equal to the present value of the investment's excess returns over the market.

However, the stochastic process driving the firm's profits implies that there always exists a positive probability that at a point in time t, $\pi_t < b$. At that point the firm is unable to pay the no-arbitrage sum b because it is illiquid, and the project's value is out of the market. Therefore, for the project to be consistently valued on the pure NPV basis, an auxiliary assumption is added concerning the existence of infinite liquidity supply at any illiquidity state (denoted by superscript l), i.e.

Assumption 2a. In all illiquidity states, $\pi_t < b$, the firm obtains an instantaneous costless injection of liquidity equal to $b - \pi_t$.

By contrast, let us consider the opposite case of *zero liquidity supply*. A paradigmatic example is given by the so-called "cash-in-advance constraint" in monetary theory (Clower, 1967; Lucas, 1980) which means that no agent can borrow liquidity in no circumstance. Consequently the end-of-time condition of zero present value of wealth is supplemented with a time-by-time cash constraint which forces money holdings into the agent's programme. Alternatively, the agent should consider the liquidation cost of non-money assets, that is to say,

Assumption 2b. In all illiquidity states, $\pi_t < b$, the firm obtains no liquidity and can only liquidate its non-money assets K.

In particular, letting $T = \inf(t \geq 0 : \pi_t \leq b)$ be the stochastic illiquidity time, i.e. the first time that $\pi_t \leq b$. Then, the project's NPV for the firm as of time 0 in this case becomes:

$$V^i(\pi; b) = E_0 \left\{ \int_0^T e^{-rt} \pi_t dt - e^{-rT} K \mid \pi_0 = \pi \right\} - I \tag{5}$$

where the superscript i indicates the illiquidity case.

By the usual dynamic programming decomposition we may split the above conditional expectation into the contribution over the infinitesimal time interval 0 to dt and the integral from dt to T. Because the investment yields a cash flow up to the time T that the project is shut down, by the no-arbitrage condition (2) the excess return from holding it, over the small time interval dt, is given by $(\pi - b)dt$ plus the capital gain $E(dV^i(\pi; b))$. Hence, in the continuation region we get the following Bellman equation:

$$rV^i(\pi; b)dt = (\pi - b)dt + E(dV^i(\pi; b)), \qquad \text{for } \pi \in [b, \infty),$$

Since π_t is driven by (1), applying *Ito's Lemma* to dV^i the asset equilibrium condition yields the following differential equation (Dixit and Pindyck, 1994, pag. 147-152):

$$\frac{1}{2}\sigma^2\pi^2 V^{i\prime\prime} - rV^i = -(\pi - b), \qquad \text{for } \pi \in [b, \infty), \tag{6}$$

with boundary conditions:

$$\lim_{\pi \to \infty} [V^i(\pi; b) - \frac{\pi - b}{r}] = 0 \tag{7}$$

$$V^i(b; b) = -K \tag{8}$$

As usual, equation (7) states that, when profits go to infinity the value of the firm must be bounded. In fact, the second term in (7) represents the discounted present value of excess returns over an infinite horizon starting from π as in (4). The boundary condition (8) stems from assumption 2b, and means that when π reaches the level b, the firm's value is given by its non-money assets (*matching value condition*). By the linearity of differential equation (6) and making use of (7), the general solution takes the form:

$$V^i(\pi; b) = \frac{\pi - b}{r} + A\pi^{\beta_2}, \qquad \text{for } \pi \in [b, \infty). \tag{9}$$

where $A < 0$ is a constant to be determined and $\beta_2 < 0$ is the negative root of the characteristic equation $\Phi(\beta) = \frac{1}{2}\sigma^2\beta(\beta-1) - r = 0$.[2] The matching value condition, that is the project's NPV for the firm at the bankruptcy threshold $\pi = b$, yields the value of the constant A:

$$A = -Kb^{-\beta_2} \tag{10}$$

The difference between the project's NPV in (4) and in (9) highlights the role of the assumptions concerning liquidity supply:

$$V^l(\pi; b) - V^i(\pi; b) = -A\pi^{\beta_2} > 0$$

The difference is positive, i.e. there is a loss of value due to truncation at the first illiquidity state T and is measured by:

$$\begin{aligned} A\pi^{\beta_2} &= -(\frac{\pi - b}{r} + K)(\frac{\pi}{b})^{\beta_2} \\ &= \left[(1-k)I - \frac{\pi}{r}\right](\frac{\pi}{b})^{\beta_2} \end{aligned}$$

Note that the valuation principles found above are independent of the nature of the firm obligation and of the recipient subject. This follows straightforwardly from Modigliani-Miller conditions. If the invested amount I has been raised by equity capital, the recipient subject are shareholders, and the firm's obligation towards shareholders are dividends, then at T the firm is unable to pay the no-arbitrage dividend b and hence shareholders wish to liquidate the firm and invest elsewhere.[3] If I has been obtained as a bank loan, b is the no-arbitrage fixed coupon the firm owes the bank. Suppose the loan has been granted under a "standard debt contract" (SDC) such that (see e.g. Freixas and Rochet, 1998, ch.4):

1) the firm pays $\min(\pi_t, b)$ at each time t,

2) bankruptcy is declared when the firm is marginally insolvent, i.e. the first time that $\pi_t < b$,

3) the bank liquidates the firm and obtains the value of the non-money assets K.

[2] The negative root is equal to $\beta_2 = \frac{1}{2} - \sqrt{(\frac{1}{2})^2 + \frac{2r}{\sigma^2}}$, with $| \frac{\partial\beta_2}{\partial\sigma} | > 0$ and $| \frac{\partial\beta_2}{\partial r} | < 0$.

[3] If liquidating the firm is costly, disinvestment will only be delayed: see e.g. McDonald-Siegel(1985).

Under SDC, our definition of illiquidity corresponds to Leland's definition of bankruptcy as "the inability of the firm to raise sufficient equity capital to meet its current debt obligation" (1994, p.1214). At T the firm cannot meet its current obligation with the bank, and cannot raise equity capital in force of the arbitrage considerations made above. At that point there is no difference for the firm between being liquidated by the bank or by the shareholders.

2.2 Discussion

We have shown that firm value depends on conditions of liquidity supply throughout the firm's life. Our contention is that the two polar cases of infinite or zero liquidity supply need be supplemented with analysis of the "normal" case of costly liquidity supply. Various reasons can be put forward. First, in reality liquidation or bankruptcy at the first illiquidity event is a rare episode; firms with liquidity problems are often refinanced by incumbent or external investors, and it is therefore of interest understanding the economic rationale behind refinancing agreements and their costs. One reason may be that liquidity problems are often perceived as the consequence of temporary unfortunate events; our model allows for a positive probability that the profit can return above the no-arbitrage dividend (coupon), and hence it may be profitable for the firm to receive a liquidity injection instead of being shut down. Second, ignoring the liquidity supply problem is particularly serious in the case of relationships with banks as explained in section 1. The liquidity service is indeed a distinctive feature of bank intermediation, banks can be viewed as specialized institutions in liquidity supply, and in practice they seldom drive their borrowers to bankruptcy on the first payment default. This is often presented as an explanation of the comparative advantage of the bank in its ability to grant liquidity when the stock market "short-termism" does not (Keynes, 1936, ch.12; Mayer, 1988; Hoshi, 1989).

The bulk of the modern theory of bank-firm relationships is concerned with the bank as "ordinary" lender, and roots the rationale of bank lending, and of the comparative advantage of bank relationships vs. market relationships, in the efficient solution of borrower-lender agency problems under costly or asymmetric information. This approach tends to overlook the bank's role as liquidity provider "on call". As we have seen above, the most typical product and tool in this approach, the SDC as an optimal solution to the borrower-lender agency problem, in fact implies that the bank makes no difference whatsoever with any other market investor as far as the liquidity-supply problem is concerned. As a matter of fact, it is now customary to address this problem following the seminal contribution by Diamond and Dybvig (1983), which is however a model of depositor-bank relationship, where the bank appears as a passive deposit institution as liquidity is simply depositors' own money that they may wish to withdraw under unexpected payment needs. Here we wish to look at the problem from the other side of the borrower-bank relationship, where the bank is an active lending institution and liquidity may or may not be created in the event of the borrower's unexpected payment needs.[4] This is matter of the next section where we shall introduce a prototypical "bank refinancing contract" (BRC) as a means whereby a bank can inject liquidity into an illiquid firm. As we shall see, this kind of contract is (or has to be) intrinsically a long-term contract, and as such it raises a number of thorny problems of which we shall examine three impinging upon the contract efficiency: cost, enforceability and "renegotiation proof-ness" (see e.g. Salaniè, 1997).

3 Firm value, costly liquidity supply and the problem of efficient bank refinancing contracts

Given the model defined in section 2.1, we wish to examine the general case in which liquidity "on call" can be obtained at a cost. For reasons that we do not formalize here, but that can be related to standard arguments of comparative efficiency in liquidity management, we assume that no private investor can supply liquidity "on call" except banks (see section 1). For concreteness, it is convenient to consider a situation where the initial investment I was funded with bank debt (e.g. a SDC) generating the fixed coupon b (see section 2.1).

In the first palce, we introduce a BRC defined as follows:

[4]In practice, a bank's borrowers are generally depositors too, and hence the Diamond-Dybvig approach may be extended to the borrower-bank relationship as far as the borrower's own deposits are concerned. We have excluded this by assuming that the firms holds no monetary assets. We think that the most interesting aspect of the problem concerns the fact that bank can supply liquidity "on call" beyond the borrower's deposits, of which overdraft facilities are the simplest instance.

Assumption 3. In any illiquidity state of the firm, $\pi_t < b$, the bank is committed to supply a compensative liquidity injection with an additional coupon payment. The firm is committed to pay the additional coupon.

In practice, if the BRC is signed, when $\pi_t < b$, the bank does not exert bankruptcy and adds the overdue amount to the overall debt charging an appropriate fee. Note that the post-BRC firm's commitment can be thought of as composed of two parts, the ordinary debt contract part paying the fixed coupon b, and the refinancing facility paying the additional coupon to be determined below.

We now need model the cost of liquidity supply. Here, we treat liquidity supply as a regulation mechanism, in the sense of Harrison (1985), of the firm's profits. The bank regulates the process π_t by means of instantaneous, infinitesimal "liquidity injections" U_t never allowing π_t to go below b. More formally, the process $\pi_t \in [0, \infty)$ is free to move as dictated by (1) as long as $\pi_t > b$, but the instant π_t crosses b from above, it is reflected at the lower barrier b. Furtermore, as to the cost of liquidity, for the sake of simplicity, we assume they are linearly increasing with the liquidity injections by the amount (see Appendix):

$$dC_t = c \times U_t$$

Owing to the introduction of a cost for liquidity, the bank and the firm have now different possible options during the project's life, which are represented in figure 1.

[**Figure 1 about here**]

We start with a verbal description of the choices tree as an introduction to the more formal treatment we shall give below. Node A corresponds to the first insolvency state of the firm (time T as defined in section 2.1). There are two options:

- AB: no liquidity injection and firm's bankruptcy
- AC: liquidity injection

AB is the usual no-liquidity case. If AC is chosen the BRC is performed: the bank refinances the firm and charges an appropriate fee, the firm stays in business and is expected to pay the original coupon b and the additional fee from that point onwards. However this is not end of the problem. After refinancing, say at time $T' > T$ and node C, the firm may find itself in such conditions that a new alternative opens up:

- CD: go on complying with the BRC
- CE: breach the BRC

If the firm chooses CE, the bank should find out the most appropriate reaction; this can be one of the two following:

- EG: exert the bankruptcy procedure
- EF: do not close the firm

We wish to stress that, contrary to the assumptions underlying the models and theories discussed in previous sections, neither liquidity injections nor bankrutpcy can be taken as unconditional pre-determined events. Both may or may not be chosen under given circumstances. Which alternative is chosen at each node, and hence the properties of the BRC, essentially depend on two factors: 1) the cost of liquidity supply; 2) the commitment technology behind the BRC.

3.1 Full commitment.

In the first place, we study a reference case where the bank-firm relationship is supported by *full commitment* by both sides.

Assumption 4. By full commitment we mean that contract conditions are perfectly verifiable at zero cost, i.e. a) each realization π_t is observable, b) the bank obtains the due amount with certainty.

In other words, if AC is chosen, also CD is alway chosen. Coming to the choices at node A, since AB is the zero liquidity-supply case in section 2, we already know that the NPV of the project is $V^i(\pi; b)$ for the firm. As to the bank, the project's NPV amounts to

$$W^i(\pi; b) = E_0 \left\{ \int_0^T e^{-rt} b dt + e^{-rT} K \mid \pi_0 = \pi \right\} - I \tag{11}$$

As T is the random first time the process π_t reaches the bound b starting from the general initial position $\pi > b$, (11) reduces to:[5]

$$\begin{aligned} W^i(\pi; b) &= \frac{b}{r}\left[1 - \left(\frac{\pi}{b}\right)^{\beta_2}\right] + K\left(\frac{\pi}{b}\right)^{\beta_2} - I \\ &= \frac{b}{r}(k-1)\left(\frac{\pi}{b}\right)^{\beta_2} \end{aligned} \tag{12}$$

It is to be noted that it must be $W^i(\pi; b) \leq 0$, for otherwise it would be more valueable for the bank to drive the firm to bankruptcy *before* the firm is unable to meet its debt obligation, which is inconsistent with mutual full commitment. Since $k \in (0, 1]$, the above condition is always satisfied for any k. In particular, the following proposition holds:

Proposition 1 *With full collateral, $k = 1$, $W^i(\pi; b) = 0$, the bank is indifferent towards bankruptcy throughout the project's life. For any $k < 1$, bankruptcy entails a loss for the bank too, due to the truncation of firms' payments.*

This is an important result which we shall use later.

Let us now consider the alternative choice AC. If it is taken, by assumption 4 both the bank and the firm will always fulfill the BRC provisions so that the project will last throughout its life; hence studying the choices at node A under full commitment is equivalent to studying the conditions of existence of the BRC.

The project's NPV for the firm under the BRC, after account is taken of the cost of liquidity, becomes:

$$\begin{aligned} V^c(\pi; b) &= E_0 \left\{ \int_0^\infty e^{-rt} [(\pi_t - b + U_t) dt - dC_t] \mid \pi_0 = \pi \right\} \\ &= \frac{\pi - b}{r} + E_0 \left\{ \int_0^\infty e^{-rt} (U_t dt - dC_t) \mid \pi_0 = \pi \right\} \end{aligned} \tag{13}$$

[5] Solution of (11) can be obtained through the usual dynamic programming decomposition. However, for a more general approach to calculate this expression see Harrison (1985, p.42).

To compute the discounted expectation in (13) we repeat the arbitrage calculation, but now with a different condition at the (reflecting) barrier b. That is, $V^c(\pi; b)$ is obtained by solving the following Bellman equation (see Appendix):

$$\frac{1}{2}\sigma^2\pi^2 V^{c\prime\prime} - rV^c = -(\pi - b) \qquad \text{for } \pi \in [b, \infty), \tag{14}$$

with boundary conditions:

$$\lim_{\pi\to\infty}[V^c(\pi; b) - \frac{\pi - b}{r}] = 0 \tag{15}$$

$$V^{c\prime}(b; b) = c \tag{16}$$

While (15) is equal to (7) in the previous case, and has the same meaning, condition (16) replaces the matching value condition (8). In fact, since liquidity is now costly, it is necessary that at each liquidity injection the marginal value of continuing the project's life does not fall below the marginal cost of liquidity (*smooth pasting condition*).[6] Again, by the linearity of the differential equation (14) and making use of (15) and (16), the general solution of (13) takes the form:

$$V^c(\pi; b) = \frac{\pi - b}{r} + B\pi^{\beta_2} \tag{17}$$

with:

$$B = \frac{(c - 1/r)b^{1-\beta_2}}{\beta_2}$$

The second term on the r.h.s. of (17) measures the NPV of total liquidity supply for the firm, i.e. the difference between the NPV of the liquidity injections $U_t dt$ and that of the additional fees dC_t:

$$B\pi^{\beta_2} = E_0\left\{\int_0^\infty e^{-rt}(U_t dt - dC_t) \mid \pi_0 = \pi\right\} \tag{18}$$

Taking into account the initial loan I and the liquidity injections U_t, the expected discounted cost of liquidity injections matches the total value of injections (see Appendix):

[6] The smooth pasting condition (16) is the first order derivative of the expected present value of a function of a Brownian motion. It does not involve any optimizing role of the barrier and requires only the continuity of the first derivative of V^c in b (Dixit, 1993, p. 27).

$$\frac{1}{r}E_0\left\{\int_0^\infty e^{-rt}dC_t \mid \pi_0 = \pi\right\} = cE_0\left\{\int_0^\infty e^{-rt}U_t dt \mid \pi_0 = \pi\right\} \tag{19}$$

We are now in a position to check whether the BRC is feasible for the firm, (i.e. which of the two choices AB, ACD is more valuable to it). For the BRC to be feasible for the firm, it is simply necessary and sufficient that the project's NPV under the BRC is not less than under the zero liquidity case, which implies the following overall participation constrain*:*

$$V^c(\pi; b) - V^i(\pi; b) = B\pi^{\beta_2} - A\pi^{\beta_2} \geq 0. \tag{20}$$

Knowing the constants A e B, the firm's participation condition becomes:

$$\frac{(c - 1/r)b^{1-\beta_2}}{\beta_2} + Kb^{-\beta_2} \geq 0.$$

Note that such a condition implies a constraint upon the determination of the marginal liquidity cost c by the bank:

$$c \leq \frac{1}{r} - \frac{K\beta_2}{b}.$$

Since $\beta_2 < 0$, $c > 1/r$ is a feasible value, hence c can *exceed the perpetual discount rate* $1/r$ by an amount proportional to the cost K and the parameter β_2. To grasp the meaning of this result, recall that K represents an exit cost to the firm which may induce it to stay in business instead of quitting immediately. Given our assumption about the collateralization degree k, i.e. $K = kI$, and given that $b = Ir$, we obtain the feasible interval for c:

$$c \leq \frac{(1 - k\beta_2)}{r} \tag{21}$$

In words, c can exceed $1/r$ up to the factor $-k\beta_2/r > 0$, which we may call the "liquidity premium". We now turn to the bank. The project's NPV for the bank under the BRC is:

$$\begin{aligned} W^c(\pi; b) &= E_0\left\{\int_0^\infty e^{-rt}[(b - U_t)dt + dC_t] \mid \pi_0 = \pi\right\} - I \qquad (22) \\ &= \frac{b}{r} - I - B\pi^{\beta_2} \\ &= -B\pi^{\beta_2} \end{aligned}$$

This is an obvious result since, as we know, $B\pi^{\beta_2}$ is the NPV of total liquidity supply for the firm, so that $-B\pi^{\beta_2}$ is the same variable valued by the bank. Note that therefore if the BRC is implemented, it implies a zero-sum *redistribution of value* between bank and firm.

To establish whether the BRC is valuable to the bank we should compare its NPV with the NPV of the alternative choice of exerting the bankruptcy procedure at node A. As we have shown previously, bankruptcy may entail a loss for the bank measured by $W^i(\pi; b) \leq 0$, hence the BRC is valuable to the bank provided that:

$$W^c(\pi; b) - W^i(\pi; b) = -B\pi^{\beta_2} + \frac{b}{r}(k-1)\left(\frac{\pi}{b}\right)^{\beta_2} \geq 0$$

After substituting the value of B, like in the case of the firm, we obtain a participation condition of the bank which implies a restriction on the marginal refinancing cost c:

$$c \geq \frac{[1 + (1-k)\beta_2]}{r} \equiv \frac{(1 - k\beta_2)}{r} + \frac{\beta_2}{r} \qquad (23)$$

Therefore, c can be set *below* $1/r$ up to the factor $(1-k)\beta_2/r < 0$ that we may call the "liquidity discount". Note that if $c \leq 1/r$, $B \geq 0$, i.e. the NPV of total liquidity supply is negative for the bank. Nonetheless, as a noteworthy consequence of the fact that the alternative to implementing the BRC is the firm's bankruptcy, and this may be a loss for the bank depending on $k < 1$, the BRC remains valuable to the bank even for values $c < 1/r$ up to the point where the loss in the NPV of total liquidity supply does not exceed the NPV of the bankruptcy loss. In other words, if owing to incomplete collateralization the firm's bankruptcy is costly to the bank, the latter is ready to pay a liquidity discount that the firm may in turn exploit by claiming for a lower c.

As a result of our inspection of the BRC participation conditions of the firm and the bank at node A in Figure 1 (i.e. from (21) and (23)), we have obtained:

Lemma 1 *The feasible set of values of the marginal refinancing cost c is :*

$$\frac{1 + (1-k)\beta_2}{r} < c \leq \frac{1 - k\beta_2}{r} \qquad (24)$$

At the upper bound of the set, $c = \frac{(1-k\beta_2)}{r}$, the bank exploits the firm's willingness to pay for liquidity and charges the liquidity premium $-k\beta_2/r$; at the lower bound of the set, $c = \frac{1+(1-k)\beta_2}{r}$, the firm exploits the bank's unwillingness to exert bankruptcy and claims for the liquidity discount $\frac{(1-k)\beta_2}{r}$. The width of the set of values of c depends on the collateralization degree k and on the absolute value of β_2, which is in turn increasing in the variance of the π_t process (i.e. $|\frac{\partial\beta_2}{\partial\sigma}| > 0$). i.e.

- A large variance process induces the bank to charge a high liquidity premium and the firm to claim for a high liquidity discount;
- A high collateral increases the firm's willingness to pay in order to stay in business, and hence it jeopardizes the firm's ability to extract a liquidity discount, while it strenghtens the bank's ability to charge a liquidity premium.
- For $k = 1$, there will be no way for the firm to induce the bank to accept a liquidity discount and c will necessarily be set above $1/r$.

By Lemma 1, for any c in the feasible set, the move ACD dominates the move AB for both the firm and the bank, hence there may be bargaining over c. We do not formalize this problem because we instead wish to stress the crucial role of the full commitment assumption. Suppose for the time being that the bank has all the bargaining power because it is on the long side of market (the bank can refuse the whole BRC to the firm whereas the firm cannot). Therefore:

Proposition 2 *Under full commitment, the bank can maximize its NPV by setting $c^* = \frac{(1-k\beta_2)}{r}$, and can extract a rent from the firm's value by the amount $-B(c^*) > 0$.*

We shall see in the next section that it is sufficient to remove the full commitment assumption to obtain a different, efficient, endogenous solution.

3.2 Observable profits but non enforceable payments.

So far we have examined situations in which once the parties have signed the BRC, both comply with the contract terms forever. Yet any borrower-lender relationship raises the fundamental question: why should the borrower pay his/her debt to the lender? The popular answer is that the borrower will pay the lender as long as the latter can monitor and audit the former, or to the extent that insolvency, leading to bankruptcy, is more costly than the debt payment. For instance, the basic model of SDC arises out of the premise that the debt payment problem exists owing to asymmetric information and costly state verification (Townsend, 1979; Diamond, 1984; Gale-Hellwig, 1985). Then it is shown that the optimal debt contract is precisely tailored to solve the problem by minimizing the lender's costs of auditing the borrower's true state, and driving the borrower into bankruptcy when he/she is truly insolvent.

Implicit in this result are two assumptions: a) auditing, albeit costly, is always effective (i.e. auditing is always sufficient for the lender to obtain the amount due); b) bankruptcy *vis-à-vis* insolvency is always viable to the lender. More recent studies have re-examined the issue after relaxing assumption a). The more general setup is one of "*non-enforceable contracts*". The idea is that, whatever the borrower's true state, and even when he/she is technically solvent, the lender's ability to bear auditing costs, or even his/her free access to information, may be of little help in the absence of the borrower's *willingness to pay.* The most typical case is when the borrower is a sovereign State (Eaton-Gersowitz, 1981; Eaton et al., 1986). However, even in the case of private relationships, there may be several reasons other than asymmetric information that may prevent debt contract provisions contingent on the borrower's state from being enforced (Bolton-Scharfstein, 1990; Hart-Moore, 1996, sec.4). One basic reason is that a lender (in civilized countries) has no private means to force the borrower to pay the amount due. Another complementary reason may be "non-verifiability" (which is indeed tantamount to infinite state verification costs), that is to say the borrower's state can be "technically" known to the two parties, but it is not possible to have it verified by an independent legal authority (e.g. the courts). In these cases, unless the borrower's willingness to pay is assumed a priori, the bankruptcy option against a declaration of insolvency must be effective at any point in time.[7] This rationale of viable borrower-lender relationships is often referred to as "threat of termination", which is generally viewed as an efficient incentive (or better disincentive) device (Stiglitz-Weiss, 1983; Allen,

[7] Bankruptcy is generally a legal procedure enforced by the courts. This does not contradict the non verifiability assumption because the courts are not requested to ascertain the borrower's ability to pay, but to act upon the borrower's unwillingness to pay.

1983; Bolton-Scharfstein, 1990; Haubrich, 1989).

This leads us to the second assumption underlying SDCs -that bankruptcy *vis-à-vis* insolvency is always viable to the lender. It is well-known to bankers that "a 1000 dollars debt is a debtor's problem, a 1,000,000 dollars debt is a creditor's problem". In early SDC models, bankruptcy is indeed a forced choice because the debt contract has a fixed deadline. However, in a borrower-lender long-run relationship, it is often the case that bankruptcy is not the most profitable (least costly) lender's choice. As seen above, our model demonstrates this point. Therefore, in all cases where termination entails a loss for the lender, the lender's "threat of termination" is weakened while the borrower's "threat of insolvency" is strengthened. Under such *mutual threat* some form of renegotiation may be more profitable. To put it differently, under the conditions under discussion, the contract may no longer be "renegotiation-proof" (Salanié, 1997), which means that under specific circumstances the borrower may be able to induce the lender to renegotiate over the contract terms instead of exerting bankruptcy. Not by chance, in the field of sovereign State debt, where there is no enforceability of payments and the lenders' termination losses are high, renegotiation is the prevailing solution when insolvency is declared. Thus, an interesting question arises: if we remove perfect commitment and shift to a context of non enforceability, can we devise efficient (i.e. minimum cost) and "renegotiation-proof" BRCs?

To address this problem we drop assumption 4 which we replace with the following:

Assumption 5. The BRC is non enforceable privately. If at any point in time the firm refuses to pay its obligation by declaring insolvency, the bank can exert the bankruptcy procedure against the firm.

STEP 1

In the first place we show that, as argued above, when the firm's bankruptcy entails a loss for the bank, its "threat of termination" is not credible. More precisely, upon the firm's declaration of insolvency at any point in time, threre exists a "fixed" coupon $z < b$ such that the bank is at least indifferent between renegotiating the contract with the new coupon z and exerting the bankruptcy procedure.[8]

[8] z non necessarily has to be fixed, we assume it constant to parallel with b, see footnote n.

At any point in time the NPV of termination for the bank, $W^t(b)$, i.e. the NPV of exerting bankruptcy at that time, is simply:

$$W^t(b) = K - I = \frac{b}{r}(k-1) \tag{25}$$

If we define $W^z(z;b)$ the project's NPV for the bank after renegotiation of the coupon z, this is given by:

$$\begin{aligned} W^z(z;b) &= E_t\left\{\int_t^\infty e^{-r(s-t)}zds\right\} - I \\ &= \frac{z-b}{r} \end{aligned} \tag{26}$$

Therefore, the bank will be indifferent between the two options provided that:

$$W^z(z;b) - W^i(b) = \frac{z-kb}{r} \geq 0$$

or, simplifying we get:

Lemma 2 *The bank's renegotiation set for a fixed coupon is:*

$$z^b \geq kb \tag{27}$$

This result shows that, as long as $k < 1$, i.e. less than full collateral, at any point in time the bank can be induced to renegotiate a new coupon $z < b$, or, in other words, the no-arbitrage coupon b cannot be enforced by "threat of termination" nor, obviously, can any additional coupon. The reason, as already stressed, is that $k < 1$, entails that termination is costly to the bank; hence it can find it profitable to obtain at least kb (forever) instead of bearing the termination loss.

STEP 2

In the second place, we examine the firm's willingness to pay. It is trivial that the firm has always an incentive to retain the bank loan and minimize the debt payment no matter whether profit is high, $\pi > b$ (see equation (3)) or low, $\pi < b$ (see equation (17)). However, it is worth giving a formal proof.

Let us suppose that at a point in time $t < T$ the firm has a profit $\pi_t = x > b$ and can choose to pay the bank a fixed coupon z such that the consequent project's NPV is:

$$\begin{aligned} V^z(x;b,z) &= E_t\left\{\int_t^{\infty} e^{-r(s-t)}(\pi_s - z)ds \mid \pi_t = x > b\right\}, \quad \text{for } t < T \qquad (28) \\ &= \frac{x-z}{r} \end{aligned}$$

Since at the same point in time the project's NPV is $V^c(x;b) = \frac{x-b}{r}$, the firm has an incentive to renegotiate if:

$$V^z(x;b,z) - V^c(x;b) = \frac{b-z}{r} \geq 0$$

or:

$$z < b \qquad (29)$$

Now let us consider the same problem at a point in time $t > T$ with profit $\pi_t = x < b$, i.e. after the refinancing part of the bank contract has been activated. Since the bank is committed to inject liquidity to fill any gap $x - b$, the firm will find it profitable to renegotiate if:

$$V^z(x;b,z) - V^c(b;b) = \frac{x-z}{r} - B(c)b^{\beta_2} \geq 0$$

or:

$$x - z - b(\frac{cr-1}{\beta_2}) \geq 0 \qquad (30)$$

To obtain the sum z that the firm is willing to pay, recall that:

- $B(c)b^{\beta_2}$ measures the NPV of total liquidity supply for the firm, which is decreasing in the marginal cost of liquidity c;
- For any value of $c < 1/r$, it follows that $B(c)b^{\beta_2} > 0$, i.e. refinancing redistributes value from the bank to the firm, and for any $c \geq 1/r$, $B(c)b^{\beta_2} \leq 0$, i.e. refinancing redistributes value from the firm to the bank;
- For any value of c within the feasible set (24), the firm prefers to stay in business under the BRC than to close. A *fortiori*, the firm prefers to pay $z < b$ than to close;
- On the other hand, the firm cannot pay a coupon that exceeds its expected profit, hence it must be $z \leq x$.

Therefore: a) for any value of $c < 1/r$ in the feasible set, the firm is willing to pay any coupon $z < x - b(\frac{cr-1}{\beta_2}) < x < b$; b) for any $c > 1/r$, the firm is willing to pay $z \leq x < b$. The difference between the two cases relates to the fact that the firm under the BRC gains value in the former and looses value in the latter; hence the firm is willing to pay less in the former than in the latter. Consequently, depending on the value of c, we can write:

Lemma 3 *The firm's renegotiation set for a fixed coupon is:*

$$z^f < \min\left[x, \;\; x - b(\frac{cr-1}{\beta_2})\right] \tag{31}$$

Note that $c > 1/r$ has an important consequence on the timing of renegotiation: this should take place as soon as the firm's profit crosses the threshold value b. Consider as an example the case that the bank is maximizing the liquidity premium by setting $c^* = \frac{1-k\beta_2}{r} > \frac{1}{r}$; then the solution to condition (30) is $z < x + kb$, which is however constrained to the subset $z \leq x$. As soon as $x = b$ the firm finds it profitable to renegotiate.

STEP 3

Our last step is to compare the renegotiation sets of z for the firm (Lemma 2) and for the bank (Lemma 3). Let us examine the case when the BRC includes a liquidity premium $c > 1/r$; hence, z^f is unconditionally the same in all states (see (30) and (31)), so that:

$$z^f < b, \quad \text{and} \quad z^b \geq kb, \tag{32}$$

Clearly, the break-even sum $z^b = kb$ for the bank lies in the renegotiation set for the firm, which implies the following proposition:

Proposition 3 *Under non-enforceability, with less than full collateral, $k < 1$, at any point in time the BRC is non "renegotiation-proof", and can be renegotiated for a fixed coupon $z = kb$. With full collateral, $k = 1$, the renegotiation set of z collapses to $z = b$, so that the BRC is indeed enforced by "threat of termination".*[9]

[9] More properly, this is one possibility, where the fixed coupon z is expected to be payed by the firm from t onward and confirmed by its actions. There may be other equilibria with lock-in at the bankruptcy state, and sustained by the bank's pessimistic expectations. For example if z is changing over time there may be no equilibria because the bank expects future reduction. However, a refinement like subgame-perfectness of the bank's "threat of termination" may eliminate such equilibria.

We have found that full collateral, i.e. no termination losses by the bank, makes its "threat of termination" effective. Hence, provided that the marginal cost of liquidity charged by the bank is in the feasible set, the firm is induced to behave to meet its obligation systematically like in the full commitment case. Consequently, the bank has the incentive to fix the marginal cost of liquidity that maximizes the rent it extracts from the firm value (Proposition 2). Under these conditions, the BRC's "renegotiation proofness" comes at the expense of efficiency. Now suppose that the collateral market value is not independent of the firm's state. A reasonable case is that $k = 1$ whenever the firm is technically solvent, $\pi_t > b$, and $k < 1$ when the firm is technically insolvent, $\pi_t < b$. In this situation the BRC fails "renegotiation proof-ness", i.e. the firm has the incentive to obtain liquidity and then to induce the bank to renegotiate a lower coupon by "threat of insolvency". Anticipating this, the bank will never sign the BRC. Should we conclude that the BRC is either inefficient or non-existent? Our key argument is that, quite the contrary, the bank-firm "mutual threat" is beneficial as it leads to an endogenous efficient solution for the value of the marginal cost of liquidity c. In other words, the efficient and "renegotiation-proof" BRC emerges from a continuous rate of hearing between the firm and the bank. Such repetition of the relationship may substitute explicit long-term contracts and provides the firm with appropriate "refinancing options"

As we have seen above, the firm should start paying for liquidity as its profit falls below b, but it has the highest incentive to breach the BRC and offer $z = kb$ to the bank soon after liquidty has been granted and profits have recovered. Yet the renegotiation set of z for the firm is a function of c (see 31).Therefore, the BRC can only be made renegotiation-proof if the bank chooses c in such a way that it drives the firm's incentive to renegotiate to zero, given its willingness to pay $z \leq x$; hence, according to (30):

$$b(\frac{cr-1}{\beta_2}) = 0, \tag{33}$$

or:

$$c = \frac{1}{r}$$

i.e., the present value of 1 euro forever. For this value of c, the NPV of total liquidity supply is driven to zero, $B = 0$,the bank extracts no rent from the firm value, and the firm value is exactly equivalent to the pure NPV of investiment. Therefore, we have found the following "efficiency propositions".

Proposition 4 *A) In order for a BRC to be "renegotiation-proof" with no private enforceability of payments, and with less than full collateral,* $k < 1$*, the marginal cost of liquidity must be* $c = 1/r$*.*

B) A BRC can be efficient only with less than full collateral and "mutual threat"

The rationale of propositions 3 and 4 is that with full collateral the bank's "threat of termination" is sufficient to enforce the BRC with the maximum liquidity rental premium above the efficient marginal cost of liquidity. With less than full collateral, the bank and the firm operate under "mutual threat". The firm's "threat of insolvency" countervails the bank's "threat of termination", so that the firm is in a position to declare insolvency and induce the bank to renegotiate. To avoid this, the bank can only refrain from charging any liquidity rental premium. Hence the bank-firm "mutual threat" is an important mechanism in the emergence of efficient firm-bank long-term relationships.

4 Conclusions

We have shown that firm value under uncertainty depends on the conditions of liquidty supply that the firm faces in possible illiquidity states. Neither infinite nor zero liquidity supply are normal situations. Costly liquidity supply "on call" granted to the borrower is a key specificity of the bank-borrower relationship. We have examined the properties of liquidity supply in the form of a refinancing contract between a valuable firm that may face temporary illiquidty and a bank. In particular, we have considered efficiency (cost-minimization) and "renegotiation proof-ness" in a context of a long-term relationship, with symmetric information but non enforceability of payments. Our main finding is that these properties of BRCs do emerge in the absence of perfect commitment as the borrower and the bank can exert *mutual threat of termination*. This condition arises whenever the bank's "threat of termination" (exert bankruptcy if the borrower is unwilling to pay) is countervailed by the borrower's "threat of insolvency" (declare insolvency and offer a renegotiation of the contract). We have found that mutual threat is effective under a simple and general condition: the loan is less than fully collateralized. In fact, if this is the case, bankruptcy entails a loss for the bank too, and hence it is willing to accept renegotiation; to avoid renegotiation, the bank has to charge no liquidity premium. An important feature of these results is that they hinge on the long-term, repeated relationship between the borrower and the bank. In this paper we have not demonstrated that efficient liquidity supply can only be granted by a bank, but as long as in our economies banks are the specialized institutions in liquidity supply, our conclusion adds weight to the idea that long-term firm-bank relationships enhance financial efficiency.

A Appendix

We define the regulation as the positive increment $d\pi_t$ to let π_t stay at b.[10] That is:

$$\tilde{\pi}_t \equiv \pi_t L_t, \quad \text{for } \tilde{\pi}_t \in [b, \infty), \tag{34}$$

or in term of the regulated process $\tilde{\pi}_t$, we get:

$$U_t = \tilde{\pi}_t - \pi_t \equiv (L_t - 1)\pi_t, \tag{35}$$

where:

- $i)$ π_t is a geometric Brownian motion, with stochastic differential as in (1);
- $ii)$ L_t is an increasing and continuous process, with $L_0 = 1$if $\pi_0 \geq b$, and $L_0 = b/\pi_0$ if $\pi_0 < b$, so that $\tilde{\pi}_0 = b$;
- $iii)$ L_t increases only when $\tilde{\pi}_t = b$.

In particular, although the process L_t may have a jump at $t = 0$ it is continuous and maintains π_t above the barrier b using the minimum amount of control, in that control takes places only when π_t would cross b from above with probability one in the absence of regulation. Applying Ito's lemma to (34), we get:

$$d\,\tilde{\pi}_t = \sigma\,\tilde{\pi}_t dW_t + d\tilde{L}_t, \quad \tilde{\pi}_0 \in [b, \infty)$$

where $d\tilde{L}_t = \tilde{\pi}_t \frac{dL_t}{L_t} \equiv \pi_t dL_t$ is the infinitesimally small level of liquidity injection from the bank to the firm. By (34), if $\tilde{\pi}_t = b$, we get $d\,\tilde{\pi}_t = 0$ and the rate of variation of L_t is equal to that of π_t to keep $\tilde{\pi}_t$ constant. Therefore, referring to $d\tilde{L}_t$, the cost of liquidity can be expressed as (Dixit, 1993, pp.42-43):

$$dC_t = cd\tilde{L}_t \equiv c\pi_t dL_t \tag{36}$$

Making use of (35) and (36) we are able to rewrite (13) as:

$$V^c(\,\tilde{\pi}_0; b) = E_0 \left\{ \int_0^\infty e^{-rt}[(\,\tilde{\pi}_t - b)dt - cd\tilde{L}_t] \mid \tilde{\pi}_0 \in [b, \infty) \right\} \tag{37}$$

[10] We use the theory of regulated stochastic process (Harrison-Taksar, 1983, Harrison, 1985). Applications of this metodology to economic problems can be found in Bentolila-Bertola (1990), Moretto-Rossini (1999) and Moretto-Valbonesi (2000).

Since $\tilde{\pi}_t$ is a Markov process in levels (Harrison, 1985, proposition 7, pp.80-81), we know that the above conditional expectation is in fact a function solely of the starting state.[11] Keeping active the dependence of V^c on $\tilde{\pi}_t$, and assuming that it is twice continuously differentiable, by Ito's lemma we get:

$$\begin{aligned} dV^c &= V^{c\prime} d\,\tilde{\pi}_t + \frac{1}{2} V^{c\prime\prime} (d\,\tilde{\pi}_t)^2 \\ &= V^{c\prime}(L_t d\pi_t + \pi_t dL_t) + \frac{1}{2} V^{c\prime\prime} L_t^2 (d\pi_t)^2 \\ &= V^{c\prime}(\sigma\,\tilde{\pi}_t dW_t + \,d\tilde{L}_t) + \frac{1}{2} V^{c\prime\prime} \tilde{\pi}_t^2 \sigma^2 dt \\ &= \frac{1}{2} V^{c\prime\prime} \tilde{\pi}_t^2 \sigma^2 dt + V^{c\prime} \sigma\,\tilde{\pi}_t dW_t + V^{c\prime} d\tilde{L}_t \end{aligned} \tag{38}$$

where we have used the property that for a finite variation process as L_t, $(dL_t)^2 = 0$. Yet, as $dL_t = 0$ except when $\tilde{\pi}_t = b$ we are able to rewrite (38) as:

$$dV^c(\tilde{\pi}_t; b) = \frac{1}{2}\sigma^2 \tilde{\pi}_t^2 V^{c\prime\prime}(\tilde{\pi}_t; b) dt + \sigma \tilde{\pi}_t V^{c\prime}(\tilde{\pi}_t; b)\, dW_t + V^{c\prime}(b; b) d\tilde{L}_t \tag{39}$$

Equation (39) is a stochastic differential equation in V^c. Now integrating by part the process $V^c e^{-rt}$ we obtain (Harrison, 1985, pag.73):

$$\begin{aligned} e^{-rt} V^c(\tilde{\pi}_t; b) = V^c(\tilde{\pi}_0; b) + & \\ & + \int_0^t e^{-rs} [\frac{1}{2}\sigma^2 \tilde{\pi}_s^2 V^{c\prime\prime}(\tilde{\pi}_s; b) - r V^c(\tilde{\pi}_s; b)] ds \\ & + \sigma \int_0^t e^{-rs} \tilde{\pi}_s V^{c\prime}(\tilde{\pi}_s; b)\, dW_s + V^{c\prime}(b; b) \int_0^t e^{-rs} d\tilde{L}_s \end{aligned} \tag{40}$$

Taking the expected value of (40) and letting $t \to \infty$, if the following conditions hold:

[11] For $\pi_0 = \pi < b$ optimal control would require that L has a jump at zero so as to ensure $\tilde{\pi}_0 = b$. In this case the integral on the right of (37) is defined to include the control cost $c\tilde{L}_0$ incurred at $t = 0$, that is (see Harrison 1985, p.102-103):

$$\int_0^\infty e^{-rt} d\tilde{L}_t \equiv \tilde{L}_0 + \int_{(0,\infty)} e^{-rt} d\tilde{L}_t$$

(a) $\lim_{u\to\infty} \Pr[T(u) > T(b) \mid \tilde{\pi}_0 \in [b,\infty)] = 0$ per $b \leq \tilde{\pi}_t < u < \infty$, where $T(u) = \inf(t \geq 0 \mid \tilde{\pi}_t = u)$ and $T(b) = \inf(t \geq 0 \mid \tilde{\pi}_t = b)$;

(b) $V^c(\tilde{\pi}_0; b)$ is bounded in $[b,\infty)$;

(c) $e^{-rt}\tilde{\pi}_t V^{c\prime}(\tilde{\pi}_t; b)$ is bounded in $[b,\infty)$;

(d) $V^{c\prime}(b; b) = c$;

(e) $\frac{1}{2}\sigma^2\tilde{\pi}_t^2 V^{c\prime\prime}(\tilde{\pi}_t; b) - rV^c(\tilde{\pi}_t; b) = -(\tilde{\pi}_t - b)$,

we obtain the expression for $V^c(\tilde{\pi}_t; b)$ indicated in (37). Condition (a) says that the probability that the regulated process $\tilde{\pi}_t$ reaches infinity before reaching some other value within $[b,\infty)$ is nul. As $\tilde{\pi}_t$ is a geometric type of process this condition is , in general, always satisfied (Karlin-Taylor, 1981, pag. 228-230). Furthermore, if condition (a) holds and $V^c(\pi; b)$ is bounded in $[b,\infty)$,then also conditions (b) and (c) hold too. Finally, its is worth noting that for $\pi_0 \geq b$, $L_0 = 1$ and then $\tilde{\pi}_0 = \pi_0 = \pi$ so that $V^c(\tilde{\pi}_0; b) = V^c(\pi; b)$. On the other hand, if $\pi_0 < b$, we get $L_0 = b/\pi_0$, so that $\tilde{\pi}_0 = b$ and $V^c(\tilde{\pi}_0; b) = V^c(b; b)$.

Finally, by the conditions $(i) - (iii)$, the policy L_t say that profits are augmented in the minimum amounts consistent with the restriction $\tilde{\pi}_t \geq b$. Further, the same conditions $(i) - (iii)$ uniquely determine L_t with the form (Harrison,1985; proposition 3, pag. 19-20):[12]

$$L_t \equiv \begin{cases} \max(1, b/\pi_0) & \text{for } t = 0 \\ \sup\limits_{0\leq v\leq t} (b/\pi_v) & \text{for } t \geq 0 \end{cases},$$

Now, we may verify that:

$$\begin{aligned} B\tilde{\pi}_0{}^{\beta_2} &= E_0\left\{\int_0^\infty e^{-rt}[(\tilde{\pi}_t - \pi_t)dt - cd\tilde{L}_t] \mid \tilde{\pi}_0 \in [b,\infty)\right\} \qquad (41) \\ &= E_0\left\{\int_0^\infty e^{-rt}[(L_t - 1)\pi_t dt - c\pi_t dL_t] \mid \tilde{\pi}_0 \in [b,\infty)\right\} \end{aligned}$$

[12] This is an applicantion of a well known result by Levy (1948) for which the process:

$$\ln\tilde{\pi}_t \equiv \ln\pi_t + \ln L_t \equiv \ln\pi_t + \sup_{0\leq v\leq t}(\ln b - \ln\pi_v)$$

has the same distribution of the “reflected Brownian process” $\mid \ln b - \ln\pi_t \mid$.

First, for each $T > 0$, the integration by parts gives:

$$\int_0^T e^{-rt}\pi_t dL_t = e^{-rT}\pi_T L_T - \pi_0 L_0 + r\int_0^T e^{-rt}\pi_t L_t dt - \int_0^T e^{-rt} L_t d\pi_t \quad (42)$$

Taking the expectation of both side and using the zero expectation property of the Brownian motion (Harrison, 1985, pag.62-63), then gives:

$$E_0 \int_0^T e^{-rt}\pi_t dL_t = \pi_T L_T E_0[e^{-rT}] + rE_0 \int_0^T e^{-rt}\pi_t L_t dt - \pi_0 L_0 \quad (43)$$

By the Strong Markov property[13] of $\tilde{\pi}_t$ it follows that $\pi_T L_T E_0[e^{-rT}] = \pi_T L_T \left(\frac{\pi}{\pi_T}\right)^{\beta_1}$ so that $\pi_T L_T \left(\frac{\pi}{\pi_T}\right)^{\beta_1} \to 0$ almost surely as $T \to \infty$. Substituting in (43) and rearranging we obtain:

$$E_0 \int_0^\infty e^{-rt}\tilde{\pi}_t dt = \frac{\pi_0 L_0}{r} + \frac{1}{r} E_0 \int_0^\infty e^{-rt}\pi_t dL_t \quad (44)$$

Therefore, rearranging we get that the expected discounted value of the firm's overdraft payments matches the total value of overdrafts:

$$\frac{1}{r} E_0 \int_0^\infty e^{-rt} dC_t = cE_0 \int_0^\infty e^{-rt} U_t dt \quad (45)$$

Finally, recalling that $\pi_0 L_0 = \pi$ if $\pi_0 \geq b$ and $\pi_0 L_0 = b$ if $\pi_0 < b$, equation (44) and (45) allow, by verification, to conclude that:

$$(cr - 1)\left[E_0 \int_0^\infty e^{-rt} U_t dt\right] \equiv B\pi^{\beta_2} \text{ or } Bb^{\beta_2}$$

[13] The Strong Markov Property of regulated Brownian motion processes stresses the fact the stochastic stopping time T and the stocastic process $\tilde{\pi}_t$ are independent (Harrison, 1985, proposition 7, pp.80-81). That is, as L_t depends only on the primitive exogenous process π_t, the Markov property extends to the endogenous regulated process $\tilde{\pi}_t$.

References

[1] Allen F. (1983), "Credit rationing and payment incentives", *Review of Economic Studies*, **50**, 639-646

[2] Anderson R., Sundaresan S. (1996), "Design and Valuation of Debt Contracts", *Review of Financial Studies*, **9**, 37-68.

[3] Bentolila S., and G. Bertola, (1990), "Firing Costs and Labour Demand: How Bad is Euroscleroris?", *Review of Economic Studies*, **57**, 381-402.

[4] Black F., Cox J. (1976), "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions", *Journal of Finance,* **31**, 351-367.

[5] Bolton P. and Scharfstein D. (1990), "A Theory of Predation Based on Agency Problems in Financial Contracting", *American Economic Review,* **80,** 93-106.

[6] Brannan M., Schwartz E. (1978), "Corporate Income Taxes, Valuation, and the Problem of Optimal Capital Structure", *Journal of Business*, **51**, 103-114.

[7] Clower R.W. (1967), "A Reconsideration of the Microfoundations of Monetary Theory",

Western Economic Journal, **4**, 80-107

[8] Diamond D.W. (1984), "Financial Intermediation and Delegated Monitoring", *Review of Economic Studies*, **51**, 393-414.

[9] Diamond D.W., Dybvig P. (1983), "Bank Runs, Deposit Insurance and Liquidity", *Journal of Political Economy,* **91**, 401-419.

[10] Dixit A., (1989), "Entry and Exit Decisions under Uncertainty", *Journal of Political Economy,* **97**, 620-638.

[11] Dixit A., (1993), *The Art of Smooth Pasting,* Harwood Academic Publishers: Chur CH.

[12] Dixit A., and R. Pindyck, (1994), *Investment under Uncertainty,* Princeton University Press: Princeton.

[13] Eaton J. and Gersowitz M. (1981), “Debt with Potential Repudiation: Theoretical and Empirical Analysis”, *Review of Economic Studies,* **48,** 289-309.

[14] Eaton J., Gersowitz M. and Stiglitz J. (1986), "On the Pure Theory of Country Risk”, *European Economic Review*, 481-513.

[15] Fama E. (1985), “What’s Different About Banks?”, *Journal of Monetary Economics*, **15,** 29-39

[16] Freixas X. and Rochet J.C. (1998), *Microeconomics of Banking*, The MIT Press: Cambridge MA.

[17] Gale D. and Hellwig M. (1985), “Incentive-compatible Debt Contracts: The One Period Problem”, *Review of Economic Studies*, **52**, 647-663.

[18] Goodhart C.A.(1989), *Money, Information and Uncertainty*, MacMillan: London, 2nd. ed.

[19] Harrison J.M., and M.T. Taksar, (1983), “Instantaneous Control of Brownian Motion”, *Mathematics and Operations Research,* **8**, 439-453.

[20] Harrison J.M., (1985), *Brownian Motions and Stochastic Flow Systems,* Wiley: New York.

[21] Haubrich J.G., (1989), “Financial Intermediation, Delegated Monitoring and Long-Term Relationships”, *Journal of Banking and Finance*, **13,** 9-20.

[22] Hellwig M. (1991), "Banking, Financial Intermediation and Corporate Finance”, in Giovannini A. and Mayer C. (eds.), *European Financial Integration*, Cambridge University Press: Cambridge, 35-63.

[23] Hicks J.R. (1967), *Critical Essays in Monetary Theory*, Clarendon Press: Oxford.

[24] Hicks J.R. (1989), *A Market Theory of Money*, Blackwell: Oxford.

[25] Karlin S., and H.M. Taylor (1981), *A Second Course in Stochastic Processes,* Academic Press: London.

[26] Keynes J.M. (1936), *The General Theory of Employment, Interest and Money*, in *The Collected Writings of John Maynard Keynes*, ed. by E. Moggridge, Macmillan: London,1973, vol.VII.

[27] Leland H. (1994), "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure", *Journal of Finance*, **49**, 1213-1252.

[28] Levy P., (1948), *Processus Stochastiques et Mouvement Brownien,* Gauthier-Villars: Paris.

[29] Lucas R.E.(1980), "Equilibirum in a Pure Currency Economy", *Economic Inquiry*, **28**, 12-33.

[30] McDonald R., and D. Siegel (1985), "Investment and Valuation of Firms when there is an Option to Shut Down", *International Economic Review,* **26**, 331-349.

[31] Mayer C. (1994), "The Assessment. Money and Banking, Theory and Evidence", *Oxford Review of Economic Policy*, **12**, pp.1-13.

[32] Mayer C. (1988), "New Issues in Corporate Finance", *European Economic Review*, **32**, 1167-1188.

[33] Mella-Barral P., Pearraudin W.R. (1997), "Strategic Debt Service", *Journal of Finance*, **52**, 531-556.

[34] Merton R. (1974), "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates", *Journal of Finance,* **29,** 449-469.

[35] Moretto M., and G. Rossini (1999), "The Efficient Assignment of the Exit Option", Department of Economics, Brown University, Working Paper n. 99-29.

[36] Moretto M., and P. Valbonesi (2000), "Option to Revoke and Regulation of Local Utilities", FEEM Nota di Lavoro n. 51.00, Milan.

[37] Perraudin W.R., Psillaki M. (1999), "Corporate Restructuring: The Impact of Loan Sales and Credit Derivatives", Birkbeck College, London, and IDEFI-CNRS, Nice, mimeo.

[38] Salanié B., (1997) *The Economics of Contracts,* MIT Press: Cambridge MA.

[39] Sharpe S.A. (1990), "Asymmetric Information, Bank Lending and Implicit Contracts: A Stylized Model of Customer Relationships", *Journal of Finance*, **45**, 1069-1087.

[40] Stiglitz J. and Weiss A. (1983), "Incentive Effects of Termination: Applications to the Credit and Labor Markets", *American Economic Review,* **73**, 912-927.

[41] Townsend R. (1979), "Optimal Contracts and Competitive Markets With Costly State Verification", *Journal of Economic Theory*, **20**, 265-93.

[42] von Thadden E.I. (1990), "Bank, Finance and Long-Term Investment", Discussion Paper no.9010, University of Basel.

Figure 1: